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Diffusive and related models of some properties of sea clutter

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Abstract

The K-distribution model for non-Gaussian statistics of scattered radiation is reviewed; the roles played by optical and radar measurements in its development and the complementary insights they provided are highlighted. A diffusive model for the K process is reviewed that makes contact with the fluctuating population and compound models inspired respectively by the optical and radar measurements. The intensity volatility of K-distributed clutter has been measured and found to differ from that characteristic of Gaussian speckle; the original diffusive model has been modified to take account of this observation. Significant problems are encountered in the computer simulation of K-distributed clutter; the diffusive models are insufficiently flexible to accommodate realistic correlation properties or to generate two-dimensional clutter images. Nonetheless the underlying phenomenology incorporated in the K model allows us to overcome these problems and practically useful simulation techniques have been devised.

1. Introduction

At the time of our collaboration, some 20 or more years ago, Peter Pusey often used to comment on the useful transfer of insights gained from scattering measurements from one wavelength regime to another. So, for example, the de Gennes narrowing phenomenon observed in neutron scattering from atomic systems has an analogue in the decay of the dynamic structure factor of a suspension of Brownian particles, probed at optical wavelengths [1], and again, the non-Gaussian statistical properties of the radiation scattered from a focused-down laser beam by a localized random medium have much in common with those of the unwanted but unavoidable sea clutter returns that plague the operation of high resolution maritime radars. Quite remarkably, equivalent models of these non-Gaussian statistics were developed, simultaneously and independently, prompted by the light scattering [2] and radar [3] studies, and the K distribution came into being. Random walks and Brownian motion also figured prominently in our discussions, as models for both particle dynamics [4] and the light scattering process itself [5]. Their descriptions in terms of Langevin and Fokker–Planck equations were

mulled over and ultimately reconciled [6], prompting an attempt to accommodate K-distributed speckle and clutter processes within such a formal framework [7]. Since the mid-1980s considerable impetus has been given to the study of Langevin and Fokker–Planck equations (in the guise of the Itô calculus) by their utility in the analysis of derivatives and other financial instruments [8]. Bearing some of these developments in mind we revisit the diffusive model, paying particular attention to the volatility ('short time diffusion constant') characterizing the evolution of the intensity of the process [9]. This suggests that a modification of the model presented in [7] is needed if we are to correctly describe the short time behaviour of the intensity. When we attempt to model sea clutter through computer simulation we find that simple diffusive models have significant shortcomings. Nonetheless, their underlying phenomenological basis is sufficient to guide us to a useful simulation technique, provided we are able to generate gamma-distributed random variables with a prescribed correlation function. The method we develop to achieve this in practice can also be adapted to an arbitrary correlated non-Gaussian process, and has found application in a variety of contexts [10, 11]. As an example, we discuss briefly how the coherent Doppler properties of sea clutter can be simulated realistically.

2. Random walks, coherent scattering and diffusion

We start our discussion with a brief review of random walks and associated processes that form the building blocks of our model. The electric field returned from N coherently illuminated scatterers can be written as a two-dimensional vector sum of the contributions $\boldsymbol{a}_k$ from the individual scatterers

$$\boldsymbol{E} = \sum_{k=1}^{N} \boldsymbol{a}_k \tag{1}$$

and be thought of as the resultant of a random walk of N steps. We assume that the $\boldsymbol{a}_k$ are independent, isotropically distributed and statistically identical; the characteristic function of the distribution of $\boldsymbol{E}$ can then be written as

$$\langle \exp(\mathrm{i}\boldsymbol{U}\cdot\boldsymbol{E})\rangle = \langle J_0(Ua)\rangle^N. \tag{2}$$

(J_0 is a zeroth order Bessel function.) To investigate the limit of a large number of scatters we scale $a \to a/\sqrt{N}$ to maintain a finite scattered energy, and find that the characteristic function tends to

$$\lim_{N\to\infty} \left\langle J_0\left(Ua/\sqrt{N}\right)\right\rangle^N = \exp(-U^2\langle a^2\rangle/4); \tag{3}$$

the corresponding distribution of the electric field has the familiar Gaussian form

$$P(E_x, E_y) = \frac{\exp(-(E_x^2 + E_y^2)/\langle a^2\rangle)}{\pi\langle a^2\rangle}. \tag{4}$$

This distribution of the field, returned from many independent scatterers, formed the basis of the classical assessments of radar performance carried out in the early 1940s [12]. It was only with the advent of high resolution radar systems, which effectively illuminated a small number of scatterers other than a possible target, that this model was found to be inadequate. As we mentioned earlier, laser measurements revealed essentially the same phenomenon at optical wavelengths.

Diffusion can be regarded as a continuous limit of a random walk process, described in terms of either a Langevin or a Fokker–Planck equation. In the former the statistical element is introduced as a Gaussian white noise term driving a diffusing particle's motion; the latter is

a partial differential equation describing the evolution in time of the probability density of the particle's position, given that at time zero. For a freely diffusing particle in one dimension we might write the Langevin equation as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t); \qquad \langle f(t_1) f(t_2)\rangle = A\delta(t_1 - t_2) \tag{5}$$

which we can integrate formally to yield

$$x(t) - x(0) = \int_0^t f(t_1)\,\mathrm{d}t_1. \tag{6}$$

The corresponding FP or diffusion equation is

$$D\frac{\partial^2 P(x,t|x(0))}{\partial x^2} = \frac{\partial P(x,t|x(0))}{\partial t}; \tag{7}$$

for these two descriptions to be equivalent we make the identification

$$A \equiv 2D \tag{8}$$

in (5). In a simple situation such as this, physical intuition allows us to make the connection between the Langevin and FP descriptions fairly unambiguously. The simple Brownian motion described by (5) forms the basis of the stochastic calculus developed by Itô. This both overcomes the mathematical difficulties presented by the white noise driving term in (5) and provides a formal framework within which more complicated Langevin (or stochastic differential) equations can be manipulated and interpreted. Transformations in the dependent variables in the SDEs and the identification of their associated FP equations can be carried through systematically, on the basis of a few relatively simple rules. Thus we define the Brownian process $B_x(t)$ as a zero-mean Gaussian with the correlation property

$$\langle B_x(t_1) B_x(t_2)\rangle = \min(t_1, t_2). \tag{9}$$

Stochastic integration with respect to the Brownian measure provided by this process is defined through the limit of a sum

$$\int_{t_a}^{t_b} \phi(x(t),t)\,\mathrm{d}B_x(t) = \lim_{n\to\infty} \sum_{k=1}^{n} \phi(x(t_{k-1}), t_{k-1})[B_x(t_k) - B_x(t_{k-1})];$$
$$t_a = t_0 \leqslant t_1 \Lambda\, t_n = t_b. \tag{10}$$

The increment $\mathrm{d}B_x(t)$ in this process in the infinitesimal time $\mathrm{d}t$ itself has a zero mean; its square, however, is identically equal to $\mathrm{d}t$, i.e.

$$(\mathrm{d}B_x(t))^2 = \mathrm{d}t, \tag{11}$$

without recourse to any ensemble averaging. This last property is a consequence of the continuity in the Brownian path. Armed with these preliminaries we re-write (5) as

$$\mathrm{d}x(t) = \sqrt{2D}\,\mathrm{d}B_x(t). \tag{12}$$

This can be supplemented by a deterministic drift term; in the case of the over-damped harmonic oscillator the SDE takes the form

$$\mathrm{d}x(t) = -\gamma x\,\mathrm{d}t + \sqrt{2D}\,\mathrm{d}B_x(t). \tag{13}$$

A more general SDE in the variable x is

$$\mathrm{d}x(t) = a(x)\,\mathrm{d}t + \sqrt{2}b(x)\,\mathrm{d}B_x(t); \tag{14}$$

here we have introduced an arbitrary drift term and allowed the diffusion 'constant' (whose square root b is referred to as a volatility in financial circles [8]) to vary with x. Solutions to

such SDEs can be developed formally by iteration, and investigated by numerical simulation. The FP equation for the conditional probability density of x, evolving subject to (14), is

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}(a(x)P(x,t)) + \frac{\partial^2}{\partial x^2}(b(x)^2 P(x,t)). \tag{15}$$

The SDE satisfied by $f = f(x)$ is given by

$$\begin{aligned} \mathrm{d}f(t) &= \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x(t) + \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(\mathrm{d}x(t))^2 = \frac{\mathrm{d}f}{\mathrm{d}x}(a(x)\,\mathrm{d}t + \sqrt{2}b(x)\,\mathrm{d}B_x(t)) + \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}b(x)^2\,\mathrm{d}t \\ &= \left(\frac{\mathrm{d}f}{\mathrm{d}x}a(x) + \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}b(x)^2\right)\mathrm{d}t + \sqrt{2}b(x)\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}B_x(t) \end{aligned} \tag{16}$$

showing how both the drift and volatility transform under a change in variable. The extension of these rules to several variables is straightforward. The interested reader can find a much fuller account of Itô calculus and related issues in Oksendal's text [13].

This formalism allows us to derive SDEs for the intensity and phase of the Gaussian speckle process obtained from the simple random walk picture. If we assume the process to be Markovian then the in phase and quadrature components of the field each satisfy an SDE analogous to (13), incorporating the effect of a carrier frequency ω

$$\begin{aligned} \mathrm{d}p_1(t) &= -\left(\frac{1}{2}p_1 + \omega p_2\right)\mathrm{d}t + \frac{1}{\sqrt{2}}\,\mathrm{d}B_1(t) \\ \mathrm{d}p_2(t) &= -\left(\frac{1}{2}p_2 - \omega p_1\right)\mathrm{d}t + \frac{1}{\sqrt{2}}\,\mathrm{d}B_2(t). \end{aligned} \tag{17}$$

If we identify the intensity and phase of the process through

$$u = p_1^2 + p_2^2, \qquad \phi = \tan^{-1}\left(\frac{p_2}{p_1}\right) \tag{18}$$

carry out the manipulations implicit in (16) and recall that the variance of a sum of independent Gaussian processes is the sum of their individual variances, we find that the SDEs for the intensity and phase are

$$\begin{aligned} \mathrm{d}u(t) &= (1 - u(t))\,\mathrm{d}t + \sqrt{2u(t)}\,\mathrm{d}B_u(t) \\ \mathrm{d}\phi(t) &= \omega\,\mathrm{d}t + \sqrt{\frac{1}{2u(t)}}\,\mathrm{d}B_\phi(t). \end{aligned} \tag{19}$$

Here B_u, B_ϕ are independent Brownian processes. In each we see that the volatility of the process depends explicitly on the intensity; 'multiplicative noise' arises quite naturally in this diffusive description of the scattered light. The FP equation satisfied by the intensity probability density is

$$\frac{\partial P(u,t)}{\partial t} = \frac{\partial^2}{\partial z^2}(uP(u,t)) + \frac{\partial}{\partial u}((u-1)P(u,t)), \tag{20}$$

noting that $\langle u \rangle = 1$.

3. Non-Gaussian light scattering and sea clutter—the K distribution

Thus far we have seen how a Gaussian speckle process emerges from the random walk picture of coherent scattering, in the limit of a large number of scatterers. In an attempt to model non-Gaussian statistics Jakeman and Pusey [14] considered the effect of fluctuations in the number of scatterers. An un-bunched Poisson distributed population was found not to induce

non-Gaussian effects in the limit of a large mean population number. A negative binomial equilibrium population

$$P(N) = \left(\frac{\mu-\lambda}{\mu}\right)^{\nu/\lambda}\left(\frac{\lambda}{\mu}\right)^{N}\frac{(\nu/\lambda)_N}{N!} \tag{21}$$

displays a characteristic bunching and is established by competing processes of birth, death and migration. These are in the rate equation satisfied by $P(N,t)$, the probability that the population takes the value N at time t:

$$\begin{aligned}\frac{\mathrm{d}P(N,t)}{\mathrm{d}t} &= \mu[(N+1)P(N+1,t) - NP(N,t)] + \nu[P(N-1,t) - P(N,t)] \\ &\quad + \lambda[(N-1)P(N-1,t) - NP(N,t)].\end{aligned} \tag{22}$$

The mean equilibrium population is given by

$$\bar{N} = \frac{\nu}{\mu-\lambda} \tag{23}$$

while its temporal fluctuations are characterized by [15]

$$\frac{\langle N(0)N(t)\rangle}{\bar{N}^2} = 1 + \left(\frac{1}{\alpha}+\frac{1}{\bar{N}}\right)\exp(-\nu t/\bar{N}). \tag{24}$$

The characteristic function of the scattered radiation now takes the form

$$\langle\exp(\mathrm{i}\boldsymbol{U}\cdot\boldsymbol{E})\rangle = \sum_{N=0}^{\infty} P(N)\langle J_0(Ua)\rangle^N = \left(\frac{\mu-\lambda}{\mu-\lambda\langle J_0(Ua)\rangle}\right)^{\frac{\nu}{\lambda}}. \tag{25}$$

By scaling the individual scattering amplitudes with the square root of the mean population we find that, as $\bar{N}\to\infty$, the characteristic function takes the form

$$\langle\exp(\mathrm{i}\boldsymbol{U}\cdot\boldsymbol{E})\rangle = \frac{1}{\left(1+\frac{\lambda\langle a^2\rangle U^2}{4\nu}\right)^{\frac{\nu}{\lambda}}} \tag{26}$$

typical of the K distribution. The infinite divisibility of the K distribution is evident from this result; this property first alerted Jakeman and Pusey [2] to the potential usefulness of this model. Rather than recovering the pdf of the scattered field amplitude by evaluating

$$P(E) = E\int_0^\infty U J_0(UE)\left(1+\frac{\lambda\langle a^2\rangle U^2}{4\nu}\right)^{-\frac{\nu}{\lambda}}\mathrm{d}U \tag{27}$$

directly, we cast it into another, potentially more revealing, form. To this end we introduce the integral representation

$$\frac{1}{\left(1+\frac{\lambda\langle a^2\rangle U^2}{4\nu}\right)^{\frac{\nu}{\lambda}}} = \frac{1}{\Gamma(\nu/\lambda)}\int_0^\infty \mathrm{d}x\, x^{\nu/\lambda-1}\exp(-x(1+\lambda\langle a^2\rangle U^2/4\nu)) \tag{28}$$

invert the order of integration and find that

$$P(E) = \frac{2E}{\Gamma(\alpha)}(\alpha/\langle a^2\rangle)^\alpha\int_0^\infty \mathrm{d}x\, x^{\alpha-1}\exp(-\alpha x/\langle a^2\rangle)\frac{\exp(-E^2/x)}{x} \qquad \alpha = \frac{\nu}{\lambda}. \tag{29}$$

The corresponding intensity distribution is

$$P(z) = \frac{1}{\Gamma(\alpha)}(\alpha/\langle a^2\rangle)^\alpha\int_0^\infty \mathrm{d}x\, x^{\alpha-1}\exp(-\alpha x/\langle a^2\rangle)\frac{\exp(-z/x)}{x}. \tag{30}$$

Thus we see that the scattered electric field is represented as a speckle process (cf (4)) whose 'local power' x is itself a gamma-distributed randomly varying quantity. This interpretation of the K distribution [3] arises quite naturally in the analysis of maritime radar data, which

typically take the form of a time series. Over short periods the Gaussian speckle statistics are evident, the process de-correlating over a time typical of transit of small scale sea surface features through the microwave wavelength (of the order 0.1 s). This speckle can be de-correlated effectively instantaneously by frequency agility, providing sufficient independent samples for the local Gaussian speckle model to be verified. The statistics of the measured local power of this process can be analysed; the gamma distribution has emerged as an excellent model in the overwhelming majority of sea clutter data studied. The modulation represented by the gamma variate may be associated with the large scale structure of the sea surface, partially resolved by the high resolution radar; its de-correlation time is typically of the order of seconds or more. A particular advantage of this compound form of the clutter model, incorporating the large separation in the timescales characteristic of the speckle and its modulation, is that it greatly facilitates the calculation of radar performance in non-Gaussian clutter. Performance in Gaussian clutter had been analysed extensively in the 1940s and 1950s and the predictions verified [12]. When high resolution maritime radars were introduced in the 1970s, and effectively vitiated any appeal to the central limit theorem to justify the assumption of Gaussian statistics, the clutter was found to be 'out of spec'. As the compound K model was found to describe this aberrant clutter quite adequately, the classic performance calculations could be redeemed, simply by averaging their results over the ubiquitous gamma distribution of local power.

4. Fokker–Planck and SDE descriptions of the K process

The compound representation of the K process is expressed in terms of continuous variables from the outset, and is perhaps the most convenient starting point for the development of its diffusive description. Equation (20) provides us with the appropriate continuous formulation of the speckle process; can we obtain a similar description of the gamma modulation from the negative binomial process? To investigate the continuous limit of the rate equation (22) we define a variable x through $N = \bar{N}x$ and expand out in inverse powers of $\bar{N}$. We can write the rate equation in terms of the PDF of x so that, on making a Taylor series expansion of the PDF and retaining terms up to second order in $1/\bar{N}$, we find that

$$\frac{1}{\bar{N}}\frac{\partial P(x,t)}{\partial t} = \frac{1}{\bar{N}}(\mu-\lambda)\frac{\partial}{\partial x}(xP(x,t)) + \frac{1}{\bar{N}^2}\left(\frac{1}{2}(\mu+\lambda)x\frac{\partial^2 P(x,t)}{\partial x^2} + (\mu-\nu+\lambda)\frac{\partial P(x,t)}{\partial x}\right). \tag{31}$$

If we are to allow $\bar{N}$ to become large, much as was done in the earlier scaling arguments that led to the K distribution, and we wish to keep the parameter ν constant, then μ, λ must tend to the same value. Thus we write

$$\frac{1}{\bar{N}}(\mu-\lambda) = \frac{\nu}{\bar{N}^2} \tag{32}$$

in the first set of terms and put $\mu = \lambda$ in the second. Finally, if we re-scale the time variable to $t \equiv t/\bar{N}$ (cf (24)) we end up with an archetypal Fokker–Planck equation

$$\frac{\partial P(x,t)}{\partial t} = \lambda\frac{\partial^2}{\partial x^2}(xP(x,t)) + \nu\frac{\partial}{\partial x}((x-1)P(x,t)) \tag{33}$$

whose equilibrium solution has the expected gamma distribution form

$$P(x) = \frac{x^{\alpha-1}\exp(-x)}{\Gamma(\alpha)}. \tag{34}$$

The equivalent SDE satisfied by x is

$$\mathrm{d}x(t) = \nu(1 - x(t))\,\mathrm{d}t + \sqrt{2\lambda x(t)}\,\mathrm{d}B_x(t). \tag{35}$$

Having assembled the constituent parts of the compound K model in diffusional form, it remains for us only to fuse them together. We have argued that the variable x is identified with the local power of the speckle process; thus we might modify the first of (19) to

$$\mathrm{d}z(t) = \left(1 - \frac{z(t)}{x(t)}\right)\mathrm{d}t + \sqrt{2z(t)}\,\mathrm{d}B_z(t). \tag{36}$$

The equilibrium density of z, x is maintained in the form

$$P(z, x) = \frac{\exp(-z/x)x^{\alpha-2}\exp(-x)}{\Gamma(\alpha)} \tag{37}$$

arising in the integrand in (30) if x satisfies the SDE

$$\mathrm{d}x(t) = A(\alpha - x(t) + z(t)/x(t)) + \sqrt{2Ax(t)}\,\mathrm{d}B_x(t). \tag{38}$$

The FP equation equivalent to (36) and (38) is

$$A\left\{\frac{\partial^2}{\partial x^2}(xP(z, x, t)) + \frac{\partial}{\partial x}((x - \alpha - z/x)P(z, x, t))\right\} + \left\{\frac{\partial^2}{\partial z^2}(zP(z, x, t)) + \frac{\partial}{\partial z}((z/x - 1)P(z, x, t))\right\} = \frac{\partial}{\partial t}P(z, x, t). \tag{39}$$

The introduction of the constant A into the model enables us to control the relative rates of de-correlation of the speckle (z) and modulation (x) processes; typically A is much less than unity. In constructing this diffusive description of the K clutter process we have been guided by the requirement that (37) is indeed the stationary solution of the FPE and that the SDEs (36) and (38) resemble (19) and (35).

Thus far, we have followed [7] quite closely. Alternatively we might represent the intensity of the K process as the product of a Rayleigh process u with unit power (cf (19) and (20)) and the modulating gamma process x (35). Using the rules of Itô calculus we can derive the SDE satisfied by this intensity as

$$\begin{aligned} &z = xu; \\ &\mathrm{d}z = x\,\mathrm{d}u + u\,\mathrm{d}x + \mathrm{d}u\,\mathrm{d}x \\ &\quad = x((1 - u)\,\mathrm{d}t + \sqrt{2u}\,\mathrm{d}B_u(t)) + u(A(\alpha - x)\,\mathrm{d}t + \sqrt{2Ax}\,\mathrm{d}B_x(t)) \\ &\quad = (x(1 - u) + uA(\alpha - x))\,\mathrm{d}t + x\sqrt{2u}\,\mathrm{d}B_u(t) + u\sqrt{2Ax}\,\mathrm{d}B_x(t) \\ &\quad = \left[-(1 + A)z + \frac{\alpha Az}{x} + x\right]\mathrm{d}t + \sqrt{2(xz + Az^2/x)}\,\mathrm{d}B_z(t). \end{aligned} \tag{40}$$

The Fokker–Planck equation describing the evolution of the joint pdf of the z and x variables now takes the form

$$\frac{\partial P(x, z, t)}{\partial t} = \frac{\partial^2}{\partial z^2}\left(\left(xz + \frac{z^2A}{x}\right)P(x, z, t)\right) - \frac{\partial}{\partial z}\left(\left(x + \frac{zA}{x} - (1 + A)z\right)P(x, z, t)\right) + A\frac{\partial^2}{\partial x^2}(xP(x, z, t)) - A\frac{\partial}{\partial x}((\alpha - x)P(x, z, t)) + 2A\frac{\partial^2}{\partial x\,\partial z}(zP(x, z, t)). \tag{41}$$

The correlation between the random components in the increments in z and x in this model is manifest in a diffusion tensor with non-diagonal elements; nonetheless, it is a simple matter to verify that (37) is the stationary solution of (41).

As we mentioned in the introduction, considerable attention has been focused on the use of SDEs in the modelling of financial instruments. One feature of this work has been the emphasis it places on the volatility-driven random terms, at the expense of deterministic drift terms. To discuss this point in any detail would take us too far afield. Here we merely comment that the celebrated Black–Scholes option price formula [8] is independent of the drift term characterizing the evolution of the asset price and that (cf (11)) an instantaneous estimate of a volatility can be made without recourse to ensemble averaging. To assess the validity of these diffusive models of the K process an analysis of the intensity volatility has recently been carried out on both radar and optical data [9]. It has been found that the volatility for the evolution in the intensity deviates from the simple root intensity form characteristic of Gaussian speckle (cf (19)); the modified result

$$(\mathrm{d}z)^2 = 2\left(xz + \frac{Az^2}{x}\right)\mathrm{d}t \tag{42}$$

captures the short time behaviour of both the optical and radar measurements. This finding suggests that the compound product model (40) is better able to describe the clutter process adequately. An anomaly detector has been devised based on the comparison of intensity volatility measurements with the form (42) and has been shown to be able to pick out small target radar returns obscured by sea clutter [9].

5. The simulation of non-Gaussian sea clutter

While FP equations such as (39) and (41), in which a linear operator generates time evolution, are quite well suited to formal manipulation, they very rarely admit closed form analytic solutions. Although (20) and (33) can be solved in terms of relatively familiar special functions [16], the solutions of (39) and (41) remain intractable. The SDE formulation of the problem perhaps provides us with an alternative, simulation route to the study of the K process. Some progress has been made along these lines [17]; in many practical circumstances, however, the diffusive formulation we have presented here is not sufficiently flexible to allow us to capture the salient features of sea clutter. Frequently we wish to simulate clutter with prescribed correlation properties; in the maritime context it is reasonable that these might have a damped, quasi-oscillatory character. Equations (36), (38) and (40) do not provide us with sufficient latitude to generate and control this sort of behaviour. An even greater problem is presented by the requirement that we generate a two-dimensional texture, with K-distributed single-point statistics and some prescribed two-point statistics. Our SDE formulation limits us quite severely to the generation of time series. Are there nonetheless features of the diffusive modelling of clutter that we can exploit in more realistic simulations?

The large separation of timescales over which the speckle and power modulation processes in (39) de-correlate allows us to effectively de-couple them, to simulate them separately and finally re-combine them multiplicatively to generate a clutter model. The product model described by (41) also suggests a simulation of this type. Because the local speckle process is Gaussian its correlation properties can be tailored quite effectively by linear techniques such as filtering or, equivalently, spectral weighting and Fourier synthesis. Thus we are left with those problems posed by the simulation of the modulating process. As mentioned earlier, there is a significant body of empirical evidence that supports the gamma model. Taken on its own the SDE (35) can furnish us only with time series data with a single exponential decaying correlation function and so is far too restrictive for our purposes. The relative ease with which we can control the correlation properties of Gaussian processes, be they time series or random fields, suggests that we consider how those correlation properties change when a zero-mean,

unit-variance Gaussian process is transformed into some non-Gaussian process by a non-linear transformation. Such a transformation can be defined by equating the cumulative distribution of the Gaussian process, evaluated at the value x taken by this process, with the cumulative distribution of the required non-Gaussian process, thus determining the latter's value η. So, if the pdf of the values η is $P_{dist}(\eta)$, we set

$$\int_{\eta}^{\infty} P_{dist}(\eta')\,\mathrm{d}\eta' = \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}\exp(-x'^2/2)\,\mathrm{d}x' = \frac{1}{2}\mathrm{erfc}\big(x/\sqrt{2}\,\big) \tag{43}$$

where, in the second equality, we have identified the complementary error function (see [18], ch 7). To generate a gamma-distributed random variable, for example, we must solve

$$\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}\exp(-x'^2/2)\,\mathrm{d}x' = \frac{1}{\Gamma(\alpha)}\int_{y}^{\infty} y'^{\alpha-1}\exp(-y')\,\mathrm{d}y'. \tag{44}$$

The complementary quantile function $Q_{dist}(\zeta)$ of the required distribution is now defined by

$$\int_{Q_{dist}(\zeta)}^{\infty} P_{dist}(\eta)\,\mathrm{d}\eta = \zeta; \tag{45}$$

using this we can write the non-linear transform that takes the input Gaussian random values into the corresponding values of the required non-Gaussian random variable as

$$\eta(x) = Q_{dist}\big(\mathrm{erfc}\big(x/\sqrt{2}\big)/2\big). \tag{46}$$

A rapidly evaluable representation of (46) is a prerequisite if this approach is to be practically feasible; [19] describes ways in which this can be constructed for the gamma distribution.

Values of η, generated from correlated values of x, will themselves be correlated; their correlation function can be expressed in the form

$$\langle\eta_1\eta_2\rangle = \int_{-\infty}^{\infty}\mathrm{d}x_1\int_{-\infty}^{\infty}\mathrm{d}x_2\,\eta(x_1)\eta(x_2)P_G(x_1, x_2, R_G), \tag{47}$$

where the joint distribution of x_1, x_2 is

$$P_G(x_1, x_2, R_G) = \frac{1}{2\pi\sqrt{1-R_G^2}}\exp(-(x_1^2 + x_2^2 - 2x_1x_2R_G)/2(1-R_G^2)). \tag{48}$$

By expanding this in terms of Hermite polynomials (see [18], ch 22), which effectively encode the factorization properties of the correlation functions of the Gaussian process, we reduce (47) to

$$\langle\eta_1\eta_2\rangle = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{R_G^n}{2^n n!}\left(\int_{-\infty}^{\infty}\mathrm{d}x\exp(-x^2)H_n(x)\,Q_{dist}(\mathrm{erfc}(x)/2)\right)^2. \tag{49}$$

Once we have evaluated the integrals we have a power series representation of the mapping between the correlation functions of the input Gaussian and output non-Gaussian processes. In a few simple cases the whole calculation can be carried through analytically; in practice recourse must almost invariably be made to numerical quadrature. The series (49) is usually found to be fairly rapidly convergent. Once this mapping has been established, it can be inverted to reveal the correlation properties of the input Gaussian process required to generate a non-Gaussian process with the specified correlation properties. This method has been applied in the controlled simulation of correlated, gamma-distributed time series and random fields with specified properties in [10]. In the work of Hopcraft *et al* [11] it is applied in the generation of correlated increments in a fractional non-Brownian motion with a power law structure function and single-point statistics resembling those of fluctuations in confined turbulent flow and the critical behaviour of a ferromagnet. Here we shall discuss briefly how coherent sea clutter

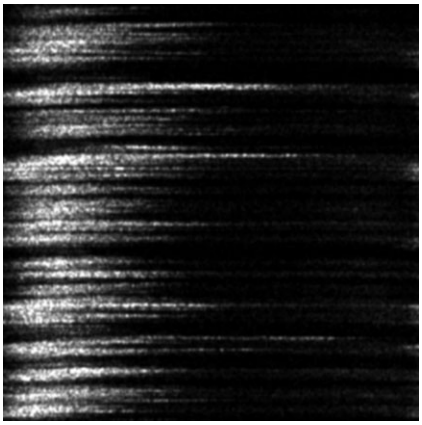

Figure 1. A range–Doppler-intensity plot of radar sea clutter taken with a 10 GHz radar at vertical polarization from a clifftop site on the south coast of England at a range of 10 km. The vertical axis is range, sampled at 1.5 m intervals over 750 m. The horizontal axis is Doppler frequency, from 0 to 250 Hz.

can be simulated; we find that the compound representation allows us to reproduce the salient features of its behaviour much better than can other methods described in the literature.

Figure 1 shows the variation with range of the Doppler spectrum of high resolution coherent sea clutter; its modulation by the partially resolved large scale structure of the sea is clearly evident. The basis of the coherent model is a complex Gaussian process whose correlation properties are related to its power (Doppler) spectrum by the Weiner–Khintchine theorem [20]. It has been suggested [21] that non-Gaussian coherent sea clutter be modelled by subjecting such a complex Gaussian process $x + \mathrm{i}y$ to a non-linear transformation that imposes the required single-point statistics. Thus, to generate coherent Weibull clutter we construct

$$\begin{aligned} x' &= x(x^2 + y^2)^{\frac{\beta-1}{2}} \\ y' &= y(x^2 + y^2)^{\frac{\beta-1}{2}} \end{aligned}, \tag{50}$$

where x and y are the real and imaginary parts of the Gaussian data. x' and y' are now the real and imaginary parts of the coherent Weibull process, which has single-point power statistics given by

$$p(\zeta > \zeta') = \exp\left[-\left(\frac{\zeta'}{\langle\zeta\rangle}\Gamma(1+\beta)\right)^{\frac{1}{\beta}}\right] \tag{51}$$

with $\zeta = \sqrt{x'^2 + y'^2}$.

It is possible to tailor the correlation properties of the input complex Gaussian to match those required of the output Weibull process reasonably well; some of the limitations on correlation properties that can be modelled in this way are discussed in [21]. However, both the input and output processes in this scheme are stationary; the Weibull clutter does not exhibit the 'breathing' modulation seen in the results in figure 1. The compound representation of the non-Gaussian clutter process allows us to remedy this defect quite straightforwardly. Rather than subjecting the complex Gaussian process itself to a non-linear transformation, we multiply it by a correlated gamma process, generated by the method we have just discussed. This product will automatically have K-distributed envelope statistics; the slowly varying gamma process imposes the 'non-stationary' breathing modulation while the relatively high frequency structure of the clutter, revealed in the Doppler spectrum, can be modelled through the spectrum assigned

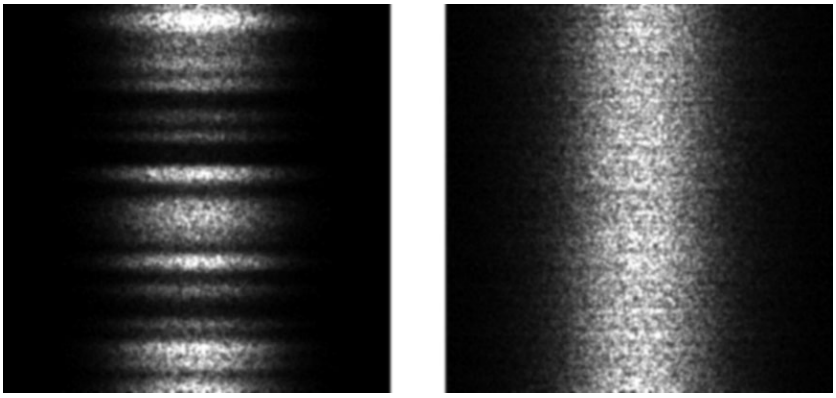

Figure 2. Coherent non-Gaussian clutter spectral time series. The left-hand image was generated using the compound method and the right using the direct non-linear transformation of a complex Gaussian. The horizontal axes are frequency and the vertical axes are range.

to the complex Gaussian process. In fact, by allowing this latter spectrum to depend on the current value of the gamma variate and the values of the shape parameter α and mean clutter power to vary with range, direction and environmental conditions, a very realistic coherent clutter simulation can be developed.

The difference between the two methods is illustrated qualitatively in figure 2, which shows the Doppler spectra of simulated clutter as a function of range. Results obtained coherent K-distributed clutter, simulated using the compound method, are shown on the left; those for coherent Weibull are simulated using the direct non-linear transformation on the left. The point amplitude statistics have been chosen to be identical (by setting $\alpha = 1/2$ and $\beta = 1$) and the average power spectra are approximately the same. The non-stationary behaviour in the image on the left and the stationary behaviour in that on the right are clearly evident.

6. Conclusions

In this paper we have reviewed some simple random walk and diffusion models for the statistics of scattered radiation, recalling how the K distribution emerged independently from the analysis of data collected in the optical and micro-wave regimes. Optical experiments highlighted a fluctuating population of scatterers as the source of non-Gaussian statistics; high resolution radar measurements partially resolved the large scale structure of the sea surface, yielding a 'locally Gaussian' return with a modulated power. Much of the success achieved in applications of the K distribution [22–24] must be ascribed to its foundation in these simple, physically motivated models that capture both the origins of the non-Gaussian statistics and the correlation properties of the processes it represents. However, the original formulations of the K process, in terms of coherent scattering, population statistics and the breakdown of the central limit theorem (in the optical case) and its compound representation (in the radar case) appear, at first sight, slightly different. The development of a Fokker–Planck description of the K process brings together these complementary pictures, identifying the gamma modulation in the compound form as a limit of the negative binomial population fluctuations and working in the continuous limit throughout. An analysis of the intensity volatility in radar and optical data suggests that the original FP formulation of the K process should be re-cast slightly to achieve better agreement with experimental observations. This has led to the development of

an anomaly detection scheme based on the measurement of 'short time diffusion constants' analogous to those determined by light scattering experiments.

We have also discussed the modelling of some salient features of coherent sea clutter by computer simulation. The compound representation of the process, as a rapidly de-correlating speckle process whose power is itself a much more slowly, but nonetheless randomly, varying quantity, modelled separately as a correlated gamma process, is particularly well suited to this purpose. Unfortunately the relatively simple dynamics captured by diffusive models are not sufficiently flexible to mimic the correlation properties of sea clutter realistically. When we consider the complexity and variability of the processes that govern the motion of the sea, this is perhaps not that surprising. However, the development of a method for the controlled simulation of gamma-distributed random variables with a specified correlation function allows us to construct realistic one- and two-dimensional clutter models.

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